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SHORT COMMUNICATION

A NECESSARY AND SUFFICIENT CONDITION FOR NONCERTAIN EXTINCTION OF A BRANCHING PROCESS IN A RANDOM ENVIRONMENT (BPRE)

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It is known that a branching process in a random environment (BPRE) which is subcritical or critical either dies with probability one or, in the trivial case, corresponds to an immortal sterile population. In the supercritical case, various conditions are known to be necessary for noncertain extinction while other conditions are known to be sufficient. In this paper, a necessary and sufficient condition for noncertain extinction of a supercritical BPRE is given. In particular, it is shown that a supercritical BPRE has noncertain extinction if and only if there exists a random truncation, depending only on the environmental sequence, such that the truncated BPRE is supercritical and such that the sequence of truncation points grows more slowly than any exponential sequence.

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1. Introduction

In this paper, we consider the Athreya–Karlin model of a branching process in a random environment (BPRE) [1]. This model for BPRE can be briefly described as follows.

Let $\boldsymbol{\xi} = (\xi_0, \xi_1, \xi_2, \dots)$ be a stationary and ergodic process with each ξ_i taking values in the space of probability distributions concentrated on the nonnegative integers. $\boldsymbol{\xi}$ is called the environmental sequence. Then a BPRE is a stochastic process $\{Z_n\}_{n=0}^{\infty}$ such that conditioned on $\boldsymbol{\xi}$, $\{Z_n\}_{n=0}^{\infty}$ is a nonhomogeneous branching process. Without loss of generality, we may assume $Z_0 \equiv 1$. Let $\phi_{\xi_n}(s)$ be the

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probability generating function for the offspring distribution ξ_n; i.e.

$$\phi_{\xi_n}(s)=\sum_{k=0}^{\infty} p_k(\xi_n)s^k, \quad 0\leq s\leq 1, \tag{1}$$

where $p_k(\xi_n)$ is the probability of a particle in the nth generation producing k offspring, conditioned on ξ_n. Let $m(\xi_n)=\phi'_{\xi_n}(1)$, the expected number of offspring of a particle in the nth generation, conditioned on ξ_n. A process satisfying $0<E(\log m(\xi_0))$ is said to be *supercritical.*

We will also assume throughout this paper that $E(\log m(\xi_0))<\infty$.

Let $q(\boldsymbol{\xi})$ denote the extinction probability; i.e.

$$q(\boldsymbol{\xi})=P(Z_n\to 0 \text{ as } n\to\infty \mid \boldsymbol{\xi}).$$

Athreya and Karlin [1] showed that $P\{q(\boldsymbol{\xi})<1\}$ equals 0 or 1. The BPRE is said to have *noncertain extinction* if $P(q(\boldsymbol{\xi})<1)=1$.

The classical Galton–Watson process, which is a special case of BPRE, is well known to have noncertain extinction if and only if the process is supercritical [3]. Smith and Wilkinson [6, 7] proved that if the environments of a supercritical BPRE are independent and identically distributed, then such a BPRE has noncertain extinction if and only if $E|\log(1-p_0(\xi_0))|<\infty$. In the general case where $\boldsymbol{\xi}$ is stationary and ergodic, Athreya and Karlin [1] proved that the above condition is sufficient for noncertain extinction. However, an example given by Tanny [9] shows that the condition is not necesary. In the same paper, Tanny also showed that the weaker condition

$$\lim_{n\to\infty}\frac{1}{n}\log(1-p_0(\xi_n))=0$$

is necessary but not sufficient.

In this paper we will prove the following theorem which provides a necessary and sufficient condition for noncertain extinction of a supercritical BPRE. We will use T throughout to denote the shift transformation, namely, $T(\xi_0,\xi_1,\ldots)$ equals $(\xi_1,\xi_2,\ldots)$.

Theorem 1. *Let* $\{Z_n\}_{n=0}^{\infty}$ *be a supercritical* BPRE *with* $E(\log m(\xi_0))<\infty$. *Then* $P(q(\boldsymbol{\xi})<1)=1$ *if and only if there exists a function* $v(\boldsymbol{\xi})$ *taking values in the positive integers such that*

$$\text{(I)}\quad E\left(\log\left(\sum_{k=0}^{v(\boldsymbol{\xi})-1} kp_k(\xi_0)+v(\boldsymbol{\xi})\sum_{k=v(\boldsymbol{\xi})}^{\infty} p_k(\xi_0)\right)\right)>0, \quad \text{and}$$

$$\text{(II)}\quad \lim_{n\to\infty}\frac{1}{n}\log v(T^n\boldsymbol{\xi})=0 \quad \textit{w.p.1.}$$

Remark 1. We may define a new BPRE $\{Z_n^v\}_{n=0}^{\infty}$ whose nth generation offspring distribution has probability generating function

$$\psi_n(s) = \psi(s; T^n\boldsymbol{\xi}; \upsilon) = \sum_{k=0}^{\upsilon(T^n\boldsymbol{\xi})-1} p_k(\xi_n)s^k + \left(\sum_{k=\upsilon(T^n\boldsymbol{\xi})}^{\infty} p_k(\xi_n)\right)s^{\upsilon(T^n\boldsymbol{\xi})}. \tag{2}$$

Note that $\psi_n(s)$ is measurable with respect to the σ field $\mathscr{F}(\boldsymbol{\xi}_n, \boldsymbol{\xi}_{n+1}, \ldots)$ and that $\{\psi_n(s)\}_{n=0}^{\infty}$ is a stationary and ergodic process [2], p. 105]. Hence $\{Z_n^{\upsilon}\}_{n=0}^{\infty}$ is a well defined BPRE called the *BPRE randomly truncated at* υ and υ is called the *random truncation.*

Remark 2. Condition (I) says that the truncated BPRE is still supercritical. Condition (II) says that the truncation points are growing more slowly than any exponential sequence. This means that we can exclude the possibility of a particle of the truncated BPRE ever producing more than an exponential number of offspring.

2. Proof of Theorem 1

We begin by proving the following sufficient condition for noncertain extinction.

Lemma 1. *Let* $\{Z_n\}_{n=0}^{\infty}$ *be a* BPRE *with environment* $\boldsymbol{\xi}$. *Suppose* $P(\phi''_{\xi_0}(1)<\infty)=1$. *If*

$$\lim_{n\to\infty} \frac{1}{n} \log^+ \phi''_{\xi_n}(1) = 0 \quad \text{w.p.1}$$

then

$$P(q(\boldsymbol{\xi})<1)=1.$$

Proof. Using Theorem 4 of [4], it follows that $P(q(\boldsymbol{\xi})<1)=1$, provided that

$$\sum_{n=0}^{\infty} \frac{\phi''_{\xi_n}(1)}{\Pi_n(\boldsymbol{\xi})(m(\xi_n))^2} < \infty \quad \text{w.p.1} \tag{3}$$

where $\prod_n(\boldsymbol{\xi}) = \prod_{k=0}^{n-1} m(\xi_k)$.

Let $0<\varepsilon<\frac{1}{4}E$ where $E = E(\log m(\xi_0))$. We note that

$$\frac{1}{n}\log \Pi_n(\boldsymbol{\xi}) = \frac{1}{n}\sum_{k=0}^{n-1} \log m(\xi_k) \tag{4}$$

which converges to E w.p.1 as $n\to\infty$ by the Birkhoff Ergodic Theorem [2]. Therefore there exists w.p.1 an integer $N=N(\varepsilon;\boldsymbol{\xi})$ such that for $n\geq N$, $\Pi_n(\boldsymbol{\xi})> e^{n(E-\varepsilon)}$. Since $0<E(\log m(\xi_0))<\infty$ and $\{\xi_i\}_{i=0}^{\infty}$ are identically distributed, it follows that

$$\lim_{n\to\infty} \frac{1}{n} \log m(\xi_n) = 0 \quad \text{w.p.1}.$$

Hence $m(\xi_n) > e^{-n\varepsilon}$ for n sufficiently large w.p.1. Also, by hypothesis,

$$\lim_{n\to\infty} \frac{1}{n} \log^+ \phi''_{\xi_n}(1) = 0 \quad \text{w.p.1.}$$

Hence $\phi''_{\xi_n}(1) < e^{n\varepsilon}$ w.p.1 for sufficiently large n. Therefore

$$\sum_{n=0}^{\infty} \frac{\phi''_{\xi_n}(1)}{\Pi_n(\boldsymbol{\xi})(m(\xi_n))^2} = 0\left(\sum_{n=0}^{\infty} \frac{e^{n\varepsilon}}{e^{n(E-\varepsilon)}(e^{-n\varepsilon})^2}\right) < \infty \quad \text{w.p.1} \tag{5}$$

and so (3) is verified. This completes the proof of the lemma.

Remark. Lemma 1 can be proved directly without Jager's result ([4]) by using a modified Kaplan–Karlin technique [5, 8].

Lemma 2. *Suppose $\{Z_n\}_{n=0}^{\infty}$ is a* BPRE *with $P(q(\boldsymbol{\xi}) < 1) = 1$. Let $\mu(\boldsymbol{\xi})$ be a random truncation of $\{Z_n\}_{n=0}^{\infty}$ and let $\psi(s;\boldsymbol{\xi};\mu)$, the truncated probability generating function, be given by* (2). *Let $m^{\mu}(\boldsymbol{\xi}) = \psi'(1;\boldsymbol{\xi};\mu)$. Then*

$$m^{\mu}(\boldsymbol{\xi}) \geq \frac{1-q(\boldsymbol{\xi})-(q(T\boldsymbol{\xi}))^{\mu(\boldsymbol{\xi})}}{1-q(T\boldsymbol{\xi})} \quad \text{w.p.1.} \tag{6}$$

Proof. Let $\psi(s) = \psi(s;\boldsymbol{\xi};\mu)$ given by (2) and consider the function $\hat{\psi}(s) = \hat{\psi}(s;\boldsymbol{\xi};\mu)$ given by

$$\hat{\psi}(s) = \frac{\psi((1-q(T\boldsymbol{\xi}))s+q(T\boldsymbol{\xi}))-\psi(q(T\boldsymbol{\xi}))}{1-\psi(q(T\boldsymbol{\xi}))}. \tag{7}$$

Since $\psi(s)$ is analytic for $-1 < s < 1$, so is $\hat{\psi}(s)$. Also it is easily seen that $\hat{\psi}^{(k)}(s) \geq 0$ for $k \geq 0$ and that $\hat{\psi}(1) = 1$. Hence $\hat{\psi}$ is a probability generating function. Furthermore, $\hat{\psi}(0) = 0$ and so $\hat{\psi}'(1) \geq 1$. But

$$\hat{\psi}'(1) = \frac{1-q(T\boldsymbol{\xi})}{1-\psi(q(T\boldsymbol{\xi}))}\psi'(1). \tag{8}$$

Therefore

$$1 \leq \frac{1-q(T\boldsymbol{\xi})}{1-\psi(q(T\boldsymbol{\xi}))} m^{\mu}(\boldsymbol{\xi}) \tag{9}$$

and so

$$m^{\mu}(\boldsymbol{\xi}) \geq \frac{1-\psi(q(T\boldsymbol{\xi}))}{1-q(T\boldsymbol{\xi})}. \tag{10}$$

Let

$$\delta = \delta(\boldsymbol{\xi}) = q(\boldsymbol{\xi}) - \psi(q(T\boldsymbol{\xi})). \tag{11}$$

By equation (13) of [1], $\phi_{\xi_0}(q(T\boldsymbol{\xi})) = q(\boldsymbol{\xi})$. Also it is clear from the definition of ψ that $\psi(s) \geq \phi_{\xi_0}(s)$ for $0 \leq s \leq 1$. Hence

$$\delta = \phi_{\xi_0}(q(T(\boldsymbol{\xi}))) - \psi(q(T\boldsymbol{\xi})) \leq 0. \tag{12}$$

Now

$$|\delta| = \psi(q(T\boldsymbol{\xi})) - \phi_{\xi_0}(q(T\boldsymbol{\xi}))$$

$$= (q(T\boldsymbol{\xi}))^{\mu(\boldsymbol{\xi})} \sum_{k=\mu(\boldsymbol{\xi})}^{\infty} p_k(\xi_0) - \sum_{k=\mu(\boldsymbol{\xi})}^{\infty} p_k(\xi_0)(q(T\boldsymbol{\xi}))^k$$

$$\leqslant (q(T\boldsymbol{\xi}))^{\mu(\boldsymbol{\xi})}. \tag{13}$$

Equations (9)–(13) together imply that

$$m^{\mu}(\boldsymbol{\xi}) \geqslant \frac{1-q(\boldsymbol{\xi})+\delta(\boldsymbol{\xi})}{1-q(T\boldsymbol{\xi})} \geqslant \frac{1-q(\boldsymbol{\xi})-(q(T\boldsymbol{\xi}))^{\mu(\boldsymbol{\xi})}}{1-q(T\xi)}. \tag{14}$$

This completes the proof of Lemma 2.

Remark. The definition of $\hat{\psi}(s)$ is similar to the one used by Tanny [9] to define the probability generating functions for the associated BPRE. Equation (7) becomes the definition for the associated probability generating function if one replaces q, the extinction probability of the BPRE $\{Z_n\}_{n=0}^{\infty}$, by q^{μ}, the extinction probability of the BPRE $\{Z_n^{\mu}\}_{n=0}^{\infty}$ (assuming $q^{\mu} \neq 1$ w.p.1).

Proof of Theorem 1. Suppose that there is a random truncation $\nu(\boldsymbol{\xi})$ satisfying conditions (I) and (II). We will prove that $P(q(\boldsymbol{\xi})<1)=1$.

Let $\{Z_n^{\nu}\}_{n=0}^{\infty}$ be the BPRE randomly truncated at ν. Then $\{Z_n^{\nu}\}_{n=0}^{\infty}$ is a supercritical BPRE and

$$P\Big(\lim_{n\to\infty} Z_n = 0 \,|\, \boldsymbol{\xi}\Big) \leqslant P\Big(\lim_{n\to\infty} Z_n^{\nu} = 0 \,|\, \boldsymbol{\xi}\Big) \quad \text{w.p.1.}$$

Hence it suffices to show that $P(\lim_{n\to\infty} Z_n^{\nu} = 0\,|\,\boldsymbol{\xi})<1$ w.p.1. To do this, we will verify that $\{Z_n^{\nu}\}_{n=0}^{\infty}$ satisfies the hypotheses of Lemma 1. By (2), the probability generating function of Z_n^{ν} conditioned on $Z_{n-1}^{\nu}=1$ and $\boldsymbol{\xi}$ satisfies

$$\psi_n''(1) = \sum_{k=0}^{\nu(T^n\boldsymbol{\xi})-1} p_k(\xi_n)k(k-1) + \Bigg(\sum_{k=\nu(T^n\boldsymbol{\xi})}^{\infty} p_k(\xi_n)\Bigg)\nu(T^n\boldsymbol{\xi})(\nu(T^n\boldsymbol{\xi})-1)$$

$$\leqslant (\nu(T^n\boldsymbol{\xi}))^2. \tag{15}$$

Therefore $\psi_n''(1)<\infty$ w.p.1 and

$$\lim_{n\to\infty} \frac{1}{n} \log^{+} \psi_n''(1) \leqslant \lim_{n\to\infty} \frac{1}{n} \log(\nu(T^{\prime}\boldsymbol{\xi}))^2 = 0 \quad \text{w.p.1} \tag{16}$$

by condition (II). Hence $\{Z_n^{\nu}\}_{n=0}^{\infty}$ satisfies the hypotheses of Lemma 1 and from the earlier remarks we conclude that $P(q(\boldsymbol{\xi})<1)=1$.

To prove the converse, we assume that $P(q(\boldsymbol{\xi})<1)=1$ and want to exhibit a random truncation ν satisfying conditions (I) and (II).

Let $0<\varepsilon<1$ be fixed. Define

$$\nu_N(\boldsymbol{\xi})=\begin{cases}1+\left[\dfrac{\log((1-q(\boldsymbol{\xi}))\varepsilon)}{\log q(T\boldsymbol{\xi})}\right] & \text{if } q(T\boldsymbol{\xi})>(1-q(\boldsymbol{\xi}))\varepsilon,\\ N & \text{if } q(T\boldsymbol{\xi})\leqslant(1-q(\boldsymbol{\xi}))\varepsilon,\end{cases}\tag{17}$$

where $[x]$ denotes the greatest integer in x and N is a positive integer. Let $m_N(\boldsymbol{\xi})=\psi'(1;\boldsymbol{\xi};\nu_N)$. We will first show that

$$m_N(\boldsymbol{\xi})\geqslant\frac{(1-\varepsilon)(1-q(\boldsymbol{\xi}))}{1-q(T\boldsymbol{\xi})}\quad\text{w.p.1.}\tag{18}$$

Consider $A_0=\{\boldsymbol{\xi};q(T\boldsymbol{\xi})>(1-q(\boldsymbol{\xi}))\varepsilon\}$. For $\boldsymbol{\xi}\in A_0$,

$$\nu_N(\boldsymbol{\xi})=1+\left[\frac{\log((1-q(\boldsymbol{\xi}))\varepsilon)}{\log q(T\boldsymbol{\xi})}\right],$$

and so

$$\begin{aligned}(q(T\boldsymbol{\xi}))^{\nu_N(\boldsymbol{\xi})}&=\exp\left(\left(1+\left[\frac{\log(1-q(\boldsymbol{\xi}))\varepsilon}{\log q(T\boldsymbol{\xi})}\right]\right)\log q(T\boldsymbol{\xi})\right)\\&\leqslant\exp(\log((1-q(\boldsymbol{\xi}))\varepsilon))=(1-q(\boldsymbol{\xi}))\varepsilon.\end{aligned}\tag{19}$$

By Lemma 2,

$$\begin{aligned}m_N(\boldsymbol{\xi})&\geqslant\frac{1-q(\boldsymbol{\xi})-(q(T\boldsymbol{\xi}))^{\nu_N(\boldsymbol{\xi})}}{1-q(T\boldsymbol{\xi})}\geqslant\frac{1-q(\boldsymbol{\xi})-(1-q(\boldsymbol{\xi}))\varepsilon}{1-q(T\boldsymbol{\xi})}\\&=\frac{(1-\varepsilon)(1-q(\boldsymbol{\xi}))}{1-q(T\boldsymbol{\xi})}.\end{aligned}\tag{20}$$

For $\boldsymbol{\xi}\in A_0^c$, $\nu_N(\boldsymbol{\xi})=N$ and $q(T\boldsymbol{\xi})\leqslant(1-q(\boldsymbol{\xi}))\varepsilon$ so

$$\begin{aligned}m_N(\boldsymbol{\xi})&\geqslant\frac{1-q(\boldsymbol{\xi})-(q(T\boldsymbol{\xi}))^N}{1-q(T\boldsymbol{\xi})}\geqslant\frac{1-q(\boldsymbol{\xi})-q(T\boldsymbol{\xi})}{1-q(T\boldsymbol{\xi})}\\&\geqslant\frac{1-q(\boldsymbol{\xi})-(1-q(\boldsymbol{\xi}))\varepsilon}{1-q(T\boldsymbol{\xi})}=\frac{(1-\varepsilon)(1-q(\boldsymbol{\xi}))}{1-q(T\boldsymbol{\xi})}.\end{aligned}\tag{21}$$

This completes the proof of (18).

Equation (18) implies

$$\begin{aligned}E(\log m_N(\boldsymbol{\xi}))&\geqslant E\left(\log\frac{(1-\varepsilon)(1-q(\boldsymbol{\xi}))}{1-q(T\boldsymbol{\xi})}\right)=\log(1-\varepsilon)+E\left(\log\frac{1-q(\boldsymbol{\xi})}{1-q(T\boldsymbol{\xi})}\right)\\&=\log(1-\varepsilon)>-\infty\end{aligned}\tag{22}$$

where the third line of (22) follows from [1, Theorem 1].

We observe that $\log m_N(\boldsymbol{\xi})$ increases to $\log m(\xi_0)$ w.p.1 as $N \to \infty$. Since

$$-\infty < E(\log m_1(\boldsymbol{\xi})) \leq E(\log m(\xi_0)) < \infty$$

and $\log m_N(\boldsymbol{\xi})$ is increasing in N, we have

$$\lim_{N\to\infty} E(\log m_N(\boldsymbol{\xi})) = E(\log m(\boldsymbol{\xi}_0))$$

by a version of the Monotone Convergence Theorem.

Take N_0 large enough so that $E(\log m_{N_0}(\boldsymbol{\xi})) > 0$. Let $\nu(\xi) = \max(N_0, \nu_{N_0}(\boldsymbol{\xi}))$. Our choice of N_0 guarantees that $\nu(\boldsymbol{\xi})$ satisfies condition (I), so to complete the proof, we need only show that condition (II) holds as well; i.e. that

$$\lim_{n\to\infty} \frac{1}{n} \log \nu(T^n\boldsymbol{\xi}) = 0 \quad \text{w.p.1}.$$

Let $A_n = \{\boldsymbol{\xi}; q(T^{n+1}\boldsymbol{\xi}) > (1-q(T^n\boldsymbol{\xi}))\varepsilon\}$. Note that if $\boldsymbol{\xi} \in A_n^c$ then $\nu(T^n\boldsymbol{\xi}) = N_0$. Letting I_B denote the indicator function of the set B, we have

$$\begin{aligned}
&\frac{1}{n}\log \nu(T^n\boldsymbol{\xi})\\
&= \frac{1}{n} I_{A_n}(\boldsymbol{\xi}) \max\left(\log N_0, \log\left(\left[\frac{\log((1-q(T^n\boldsymbol{\xi}))\varepsilon)}{\log q(T^{n+1}\boldsymbol{\xi})}\right]+1\right)\right) + \frac{1}{n} I_{A_n^c}(\boldsymbol{\xi}) \log N_0\\
&\leq \frac{1}{n} I_{A_n}(\boldsymbol{\xi}) \log\left(\frac{2\log((1-q(T^n\boldsymbol{\xi}))\varepsilon}{\log q(T^{n+1}\boldsymbol{\xi})}\right) + \frac{1}{n} I_{A_n}(\boldsymbol{\xi}) \log N_0 + \frac{1}{n} I_{A_n^c}(\boldsymbol{\xi}) \log N_0\\
&\leq \frac{1}{n} I_{A_n}(\boldsymbol{\xi}) \log\left(\frac{\log(1-q(T^n\boldsymbol{\xi}))\varepsilon)}{\log q(T^{n+1}\boldsymbol{\xi})}\right) + \frac{1}{n}(\log N_0 + \log 2).
\end{aligned} \tag{23}$$

Since N_0 is a fixed integer, it suffices to show that

$$\lim_{n\to\infty} \frac{1}{n} I_{A_n}(\boldsymbol{\xi}) \log\left(\frac{\log((1-q(T^n\boldsymbol{\xi}))\varepsilon)}{\log q(T^{n+1}\boldsymbol{\xi})}\right) = 0 \quad \text{w.p.1}. \tag{24}$$

Clearely we may restrict our attention to $A = \{\boldsymbol{\xi}: \boldsymbol{\xi} \in A_n$ for infinitely many $n\}$ and we need only take the limit along subsequences $n' = n'(\boldsymbol{\xi})$ for which $\boldsymbol{\xi} \in A_{n'}$ for all n'.

Now

$$\begin{aligned}
&\frac{1}{n'} I_{A_{n'}}(\boldsymbol{\xi}) \log\left(\frac{\log((1-q(T^{n'}\boldsymbol{\xi}))\varepsilon)}{\log q(T^{n'+1}\boldsymbol{\xi})}\right)\\
&= \frac{1}{n'} \log|\log((1-q(T^{n'}\boldsymbol{\xi}))\varepsilon)| - \frac{1}{n'} \log|\log q(T^{n'+1}\boldsymbol{\xi})|.
\end{aligned} \tag{25}$$

Since $\boldsymbol{\xi} \in A_{n'}$,

$$|\log(q(T^{n'+1}\boldsymbol{\xi}))| < |\log(((1-q(T^{n'}\boldsymbol{\xi}))\varepsilon)|. \tag{26}$$

Our proof is complete if we show that

$$\limsup_{n\to\infty} \frac{1}{n} \log|\log((1-q(T^n\boldsymbol{\xi}))\varepsilon)| \le 0 \quad \text{w.p.1.} \tag{27}$$

and that

$$\liminf_{n\to\infty} \frac{1}{n} \log|\log q(T^n\boldsymbol{\xi})| \ge 0. \quad \text{w.p.1.} \tag{28}$$

For if (27) and (28) hold then it follows immediately that the right hand side of (25) is 0, so (24) holds.

Considering (27) first, we have

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \log|\log((1-q(T^n\boldsymbol{\xi}))\varepsilon)| &\le \limsup_{n\to\infty} \frac{1}{n}|\log((1-q(T^n\boldsymbol{\xi}))\varepsilon)| \\ &\le \limsup_{n\to\infty} \frac{1}{n}|\log(1-q(T^n\boldsymbol{\xi}))| + \lim_{n\to\infty} \frac{1}{n}|\log\varepsilon| = 0\end{aligned} \tag{29}$$

by [9, Proposition 5.7] and so equation (27) holds.

Now, if equation (28) fails, then there must exist w.p.1 a sequence $a_k = a_k(\boldsymbol{\xi})$ of integers such that $a_k \uparrow \infty$ and

$$\lim_{k\to\infty} \frac{1}{a_k} \log|\log q(T^{a_k}\boldsymbol{\xi})| < 0. \tag{30}$$

Hence

$$\lim_{k\to\infty} \log|\log q(T^{a_k}\boldsymbol{\xi})| = -\infty \quad \text{w.p.1,} \tag{31}$$

and so

$$\lim_{k\to\infty} \log q(T^{a_k}\boldsymbol{\xi}) = 0 \quad \text{w.p.1.} \tag{32}$$

Therefore

$$\lim_{k\to\infty} \frac{|\log q(T^{a_k}\boldsymbol{\xi})|}{1-q(T^{a_k}\boldsymbol{\xi})} = 1 \quad \text{w.p.1.} \tag{33}$$

(Note that (33) follows from the expansion of $\log(1+x)$ for $|x|<1$.)

Hence

$$\lim_{k\to\infty} \frac{1}{a_k} \log|\log q(T^{a_k}\boldsymbol{\xi})| = \lim_{k\to\infty} \frac{1}{a_k} \log(1-q(T^{a_k}\boldsymbol{\xi})) = 0 \tag{34}$$

by [9, p. 111]. But this contradicts (30). Therefore (28) holds.

Therefore $\nu(\boldsymbol{\xi})$ satisfies conditions (I) and (II), and our proof is complete.

Remark. Tanny [9] gives a sufficient condition for noncertain extinction of BPRE. It is still unknown if that condition is also necessary, and hence equivalent to the condition presented in this paper.

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